Characterizing Genuine Multisite Entanglement in Isotropic Spin Lattices

Himadri Shekhar Dhar[1], Aditi Sen(De)[2], and Ujjwal Sen[2]
[1]*School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India*
[2]*Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India*

We consider a class of large superposed states, obtained from dimer coverings on spin-1/2 isotropic lattices. We show that they are genuinely multiparty entangled, irrespective of the geometry and dimension of the isotropic lattice. We then present an efficient method to characterize the genuine multisite entanglement in the case of isotropic square spin-1/2 lattices, with short-range dimer coverings. We use this iterative analytical method to calculate the multisite entanglement of finite-sized lattices, which, through finite-size scaling, enables us to obtain the estimate of the multisite entanglement of the infinite square lattice. The method can be a useful tool to investigate other single- and multisite properties of such states.

I. INTRODUCTION

Genuine multisite entanglement is an important resource in quantum information protocols and is known to offer significant advantage in quantum tasks in comparison to bipartite entanglement [1]. In particular, it is the basic ingredient in measurement-based quantum computation [2], and is beneficial in various quantum communication protocols [3], including secret sharing [4] (cf. [5]). Apart from the conventional information tasks, the study of multisite entanglement turns out to be important in understanding many-body phenomena like quantum phase transitions [6] and in biological mechanisms to understand transport properties in the evolution of photsynthetic complexes [7]. Although bipartite entanglement in the case of two spin-1/2 particles is rather well-understood, the situation is quite different in the case of classification and quantification of entanglement in higher dimensions as well as in multiparty systems. The fact that many-particle systems can have different types of useful entanglement, depending on the particular information processing protocols under study, makes the quantification a formidable task. While there are several multiparty entanglement measures known in the literature [1, 8], it is in general difficult to compute them. However, for pure multisite states, it is possible to use the generalized geometric measure [9], which is a computable measure of genuine multisite entanglement.

The isotropic spin-1/2 lattice states, with short-range dimer coverings, are of considerable interest in investigating high-temperature superconductivity [10, 11], and critical phenomena in many body systems [12–14]. These short-range dimer states can be efficiently simulated in laboratories using atoms in optical lattices [15] or in cavities using interacting photons [16]. The large number of superpositions of the nearest neighbor dimer coverings even in moderately large two-dimensional (2D) lattices, makes the computation of both bipartite and multisite entanglement of such spin states an arduous task.

In this article, we consider quantum states, with dimer coverings, of an isotropic spin-1/2 lattice. We prove that such spin-1/2 rotationally invariant states are always genuinely multisite entangled, irrespective of the lattice geometry and dimension. We present a method to analytically calculate the genuine multipartite entanglement, viz. the generalized geometric measure [9], of these quantum spin states, with short-ranged dimer coverings, on a square lattice with an arbitrary number of sites. The method enables us to calculate the genuine multisite entanglement for moderately large lattices, and perform finite-size scaling [17] to predict the measure for the infinite two-dimensional square lattice.

The paper is organized as follows. The succeeding section provides a formal definition of the spin state that we consider. Section III contains the theorem proving that the spin state is genuinely multiparty entangled. In section IV, we present a method to evaluate properties of such states on two-dimensional square lattices. In particular, a finite-size scaling is presented in Section IV C. We conclude with some discussions in Section V.

II. THE SPIN STATE

An isotropic spin-1/2 system, with dimer coverings, is conveniently defined by using a *bipartite lattice*. A bipartite lattice is one which is divided into two sublattices, A and B, such that each site of sublattice A has only sites of sublattice B as its nearest neighbors. Moreover, each lattice site is occupied by a spin-1/2 particle. The dimer-covering state on such a bipartite lattice can be written in the form [18],

$$|\psi\rangle = \sum_k h_k(i_a, i_b)|(a_1, b_1), (a_2, b_2)...(a_N, b_N)\rangle_k, \quad (1)$$

where each k refers to a single dimer covering $(\{|a_i, b_i\rangle\}_i)$ on the entire lattice with N spins in each sublattice. Here, the "covering function" $h_k(i_a, i_b)$ is a function that is isotropic over the lattice sites $i_a \in A, i_b \in B$.

The above states, with short-range dimer coverings are often referred as resonating valence bond (RVB) states [11, 19]. In quantum spin ladders, these states have been argued to be possible ground state approximations of certain Heisenberg spin-1/2 systems. Spin systems, such as RVB ladders, have been extensively used to study entanglement properties [20, 21].

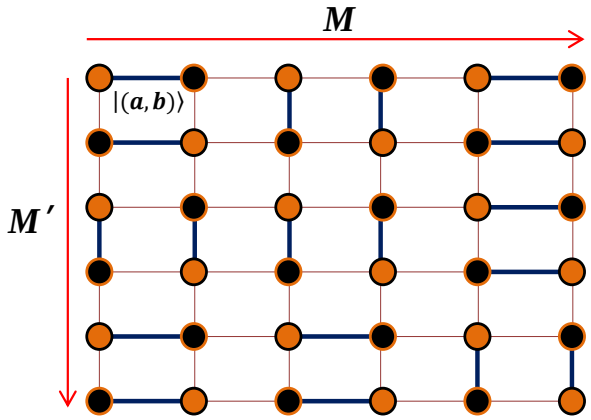

FIG. 1. (Color online.) A square spin-1/2 bipartite lattice with nearest-neighbor dimer coverings. Here, M, M'=6. The sublattice A consists of the sites denoted by the lighter circles with darker borders, while those in B has darker circles with lighter borders. The thick solid lines show the nearest-neighbor dimer states ($|(a_i, b_j)\rangle$) from a site in sublattice A to another in B. The figure represents a possible dimer covering. The final state is the superposition of all such dimer coverings.

III. CHARACTERIZING MULTISITE ENTANGLEMENT

A multiparty pure state is said to be genuinely multiparty entangled if it is entangled across every possible bipartition of the system. Examples of genuinely mulitisite entangled states include the Greenberger-Horne-Zeilinger [22] and the W [23, 24] states. The following theorem shows that the state in Eq. (1) is genuinely multisite entangled.

Theorem: *The pure state formed by superpositions of dimer coverings, is genuinely multisite entangled, for all isotropic spin-1/2 lattices of arbitrary dimensions which are periodic or infinite in all directions and all covering functions that are isotropic over the lattice.*

`Proof.` The superposition state of the spin-1/2 lattice consisting of $2N$ particles, in Eq. (1), is a pure state. To prove that this superposed state is genuinely multisite entangled, we are required to prove that the partial density matrix of the state across any bipartition cannot be pure. In other words, the density matrix of any p spins, formed by tracing the remaining $2N - p$ spins, is always mixed, and hence entangled to the rest. We conveniently divide the proof into the two cases where the number of spins (a) is finite in at least one part of the bipartition and (b) is infinite in both the parts.

a) Finite case: For a rotationally invariant state, like the dimer-covered spin-1/2 state under consideration, it is known that the partial density matrix of an arbitrary number of spins is also rotationally invariant. Moreover, for an odd number of spin-1/2 particles, there is no pure quantum state that is rotationally invariant. Hence any odd bipartition of the system is always entangled to the rest of the system. For example, any single-site density matrix is $\frac{1}{2}\mathcal{I}$, where $\mathcal{I}$ is the 2×2 identity matrix, and therefore is maximally entangled to the rest of the lattice.

Let us now consider the case of a bipartition with an even (finite) number of spins in one part. Consider any set X of an even number of sites. Let the partial density matrix of these sites in X, corresponding to the state $|\psi\rangle$, be $\rho^{(X)}$. Let us assume that $\rho^{(X)}$ is pure, which would imply that $|\psi\rangle$ is separable, contrary to the statement of the theorem. Let $X = X' \cup c$, where c contains an arbitrary but fixed odd number of sites $< |X|$. ($|S|$ denotes the cardinality of the set S.) In particular, $|c|$ can be unity. For an isotropic lattice, we can always find another equivalent set of spins, $Y' \cup c$, such that $|Y'| = |X'|$, which is again pure (by the assumption). The strong subadditivity of von Neumann entropy [25] implies

$$S(\rho^{(X')}) + S(\rho^{(Y')}) \leq S(\rho^{(X' \cup c)}) + S(\rho^{(Y' \cup c)}), \qquad (2)$$

where $S(\cdot)$ denotes the von Neumann entropy of its argument. Since $\rho^{(X)}$ and $\rho^{(Y)}$ are pure, and since von Neumann entropy is nonnegative, we have $S(\rho^{(X')}) + S(\rho^{(Y')}) = 0$, which in turn implies that $\rho^{(X')}$ and $\rho^{(Y')}$ are pure. This immediately implies that $\rho^{(c)}$ is pure. This is a contradiction as $|c|$ is pure. This part of the proof was partially presented in Ref. [20].

b) Infinite case: Let us begin with the case of an infinite 2D square lattice partitioned into two half-planes by an infinite horizontal line. Let us assume that the reduced states are pure, contrary to the statement of the theorem. Let H_P denote the sites in one such half plane. Consider now the set $H_P \cup L_H$, where L_H is the infinite horizontal strip of sites with single-site width, and which is directly adjacent to H_P. By isotropy, if $\rho^{(H_P)}$ is pure, $\rho^{(H_P \cup L_H)}$ is also pure. Consequently, we can again use strong subadditivity to show that $\rho^{(L_H)}$ is pure. Now if $\rho^{(L_H)}$ is pure, by isotropy, $\rho^{(L_V^{(p_1,p_2)})}$ is also pure, where $L_V^{(p_1,p_2)}$ is the infinite vertical strip of sites with single-site width, and which is obtained from L_H by rotating it by $\pi/2$ radians around the site with Cartesian coordinates (p_1, p_2). Again using strong subadditivity, we find that $\rho^{(p_1,p_2)}$ is pure, which is a contradiction. A similar proof works for boundaries which are not straight lines.

One needs a separate proof for the case when we want to prove that the infinite horizontal strip L_{H^r} having an width of r sites is not a product with the remaining portion of the lattice. Assume, if possible, that $\rho^{(L_{H^r})}$ is pure. Then, by isotropy, $\rho^{(L_{V^r}^{(p_1,p_2)})}$ is also pure, where $L_{V^r}^{(p_1,p_2)}$ is obtained from L_{H^r} by a $\pi/2$ rotation around (p_1, p_2). Again applying strong subadditivity, we have that the reduced state of the $2r$ spins in $L_{H^r} \cap L_{V^r}^{(p_1,p_2)}$ is pure, which is a contradiction (by the (a) part of the proof). Similar proofs are possible for other infinite strips.

Just like the proof of the (a) part, the (b) part also works, with suitable modifications, for arbitrary isotropic

lattices of arbitrary dimensions, e.g. the triangular and hexagonal lattices in 2D and the cubic lattice. ■

IV. QUANTIFYING MULTISITE ENTANGLEMENT

In the preceding section, we have demonstrated that the dimer-covering superposition on an arbitrary isotropic lattice is genuinely multisite entangled. It is interesting to quantify the amount of genuine multipartite entanglement present in such states. This is important because the qualitative answer in the preceding section indicates only the non-vanishing of the entanglement content. However, such an entanglement content can asymptotically graze to zero with increasing system-size. Below we show that this is not the case. Specifically, we provide an analytical method to calculate genuine multipartite entanglement of such dimer-covered superposition states and then show that the state under consideration possess a relatively high genuine multiparty entanglement for arbitrarily large system-size.

To quantify the genuine multisite entanglement in an isotropic spin-1/2 lattice, with dimer coverings, we use the generalized geometric measure (GGM) [9]. The GGM of an R-party quantum state is the optimized fidelity distance of the state $|\psi_R\rangle$ from the set of all states that are not genuinely multiparty entangled. More specifically, the GGM ($G(|\psi_R\rangle)$) can be calculated as

$$G(|\psi_R\rangle = 1 - \Lambda^2_{max}(|\psi_R\rangle) \tag{3}$$

where $\Lambda_{\max}(|\psi_R\rangle) = \max|\langle\chi|\psi_R\rangle|$. $|\chi\rangle$ is an R-party quantum state with no genuine multisite entanglement. The maximization is over all possible $|\chi\rangle$ states.

The GGM [9] of an R-party pure state can be efficiently calculated by using the relation

$$G(|\psi_R\rangle) = 1-\max\{\lambda^2_{K:L}|K\cup L = \{A_1,\ldots,A_R\}, K\cap L = \emptyset\}, \tag{4}$$

where $\lambda_{K:L}$ is the maximal Schmidt coefficients in all possible bipartite splits $K:L$ of $|\psi_R\rangle$.

The computation of the GGM, therefore, depends on the efficient generation of arbitrary reduced density matrices across all possible bipartitions of the spin system. Certain properties like two-point correlations, energy, and partition function, of such large superpositions, if found to be ground states of a Hamiltonian, can be evaluated in some cases by using approximate methods like mean field approximations and renormalization techniques [26]. We propose an iterative method, the *density matrix recursion method* (DMRM) for 2D square spin-1/2 lattices, with nearest neighbor dimer coverings (i.e. $h_k(i_a,i_b) = 1$ for nearest neighbors, and vanishing otherwise), by which the limitations in computing all possible density matrices and its correlation properties of such lattice states can be overcome. The method proceeds by deriving an algebraic recursion relation between the quantum states of small-sized spin systems that can be exactly computed.

We now apply the DMRM to two kinds of 2D systems: a) The "perfect" square lattices with an equal even number of sites on the horizontal and vertical sides, and b) the "imperfect" square lattices having an even number of sites on the horizontal side but an odd number on the vertical one. It is important to note that the total number of spin sites is always even. As we see later, the difference between the behavior of GGM of perfect and imperfect square lattices significantly reduces as the lattice size is increased.

A. Perfect Square Lattice

We consider a spin-1/2 square lattice with M spins along the horizontal and M' spins in the vertical sides, such that $M.M'$ $(=2N)$ is the total number of spins. For a perfect lattice with even M $(M' = M)$, the spin state of the system, which we rename as $|M,M\rangle$, can be generated by using a recursion of two smaller sized imperfect spin states $|M-1,M\rangle$ and $|M-2,M\rangle$. We drop the second M $(=M')$ from the states, as it remains unchanged. For a perfect lattice system, with $M = M' = \mathcal{N}+2$, the recursion relation, with open boundary condition, can be written as

$$\begin{aligned}|\mathcal{N}+2\rangle &= |\mathcal{N}+1\rangle|1\rangle_{n+2} + |\mathcal{N}\rangle|\bar{2}\rangle_{n+1,n+2}\\ &= |\mathcal{N}\rangle|2\rangle_{n+1,n+2} + |\mathcal{N}-1\rangle|\bar{2}\rangle_{n,n+1}|1\rangle_{n+2},\end{aligned} \tag{5}$$

where the subscripts correspond to the numbering of the sites on the horizontal side on the lattice. For our analysis, we consider a spin-1/2 square lattice state, with nearest neighbor dimers, that is periodic along the horizontal axis [27] (cf. [28]). Here, $|1\rangle_i$ refers to the state corresponding to $(M,M') = (1,\mathcal{N}+2)$ at the column i, and $|2\rangle_{i,j}$ refers to the state with $(M,M') = (2,\mathcal{N}+2)$ at the columns i and j. The state $|\bar{2}\rangle_{i,j}$ is $|2\rangle_{i,j} - |1\rangle_i|1\rangle_j$. The subtraction in the term $|\bar{2}\rangle_{i,j}$ removes a repetition in the recursion. The recursion can be extended to states with the periodic boundary condition as

$$|\mathcal{N}+2\rangle_P = |\mathcal{N}+2\rangle_{1,n+2} + |\mathcal{N}\rangle_{2,n+1}|\bar{2}\rangle_{n+2,1}. \tag{6}$$

The states without subscript P implies non-periodic states. The recursion for the corresponding density matrix is given by

$$\begin{aligned}\rho_P^{(\mathcal{N}+2)} &= \rho^{(\mathcal{N}+2)} + |\mathcal{N}+2\rangle\langle\mathcal{N}|_{2,n+1}\langle\bar{2}|_{n+2,1}\\ &+|\mathcal{N}\rangle_{2,n+1}|\bar{2}\rangle_{1,n+2}\langle\mathcal{N}+2||\mathcal{N}\rangle\langle\mathcal{N}|_{(2,n+1)}|\bar{2}\rangle\langle\bar{2}|_{(1,n+2)}\end{aligned} \tag{7}$$

where $\rho^{(\mathcal{N}+2)} = |\mathcal{N}+2\rangle\langle\mathcal{N}+2|$ is the density matrix for the non-periodic recursion. Hence using (7), we can generate the reduced density matrices across different partitions that can be used to find the behavior of GGM in the system. Tracing over all the spins except the spins

at columns $n+1$ and $n+2$, we obtain the corresponding reduced density matrix in the case of open boundary condition as

$$\rho^{(2)}_{(n+1,n+2)} = \mathcal{Z}_{\mathcal{N}}|2\rangle\langle 2|_{(n+1,n+2)} + \mathcal{Z}_{\mathcal{N}-1}\bar{\rho}_{n+1}\otimes|1\rangle\langle 1|_{(n+2)} + (|2\rangle_{n+1,n+2}\langle 1|_{n+2}\langle\xi_{\mathcal{N}}|_{n+1} + h.c.), \quad (8)$$

where $\mathcal{Z}_{\mathcal{N}} = \langle\mathcal{N}|\mathcal{N}\rangle$, $\bar{\rho}_{n+1} = \mathrm{tr}_n[|\bar{2}\rangle\langle\bar{2}|_{(n,n+1)}]$, and $\langle\xi_{\mathcal{N}}|_{n+1} = \langle\bar{2}|_{n,n+1}\langle\mathcal{N}-1|\mathcal{N}\rangle$. The reduced state in the case of periodic boundary condition is

$$\begin{aligned}\rho^{(2)}_{P(n+1,n+2)} &= \rho^{(2)}_{n+1,n+2} + \mathrm{tr}_{1..n}[|\mathcal{N}\rangle\langle\mathcal{N}|_{(2,n+1)}|\bar{2}\rangle\langle\bar{2}|_{(1,n+2)} \\ &\quad + (|\mathcal{N}\rangle_{2,n+1}|\bar{2}\rangle_{1,n+2}\langle\mathcal{N}+2| + h.c.)] \\ &= \rho^{(2)}_{n+1,n+2} + \beta^{(2)}_{1(n+1,n+2)} + \beta^{(2)}_{2(n+1,n+2)}. \quad (9)\end{aligned}$$

Here,

$$\beta^{(2)}_{1(n+1,n+2)} = \mathcal{Z}_{\mathcal{N}-1}|1\rangle\langle 1|_{(n+1)}\otimes\bar{\rho}_{n+2}\mathcal{Z}_{\mathcal{N}-2}\bar{\rho}_{n+1}\otimes\bar{\rho}_{n+2} + \mathcal{Z}_{\mathcal{N}-2}\bar{\rho}_{n+1}\otimes\bar{\rho}_{n+2}(|1\rangle\langle\xi_{\mathcal{N}-1}|_{(n+1)}\otimes\bar{\rho}_{n+2} + h.c.), \quad (10)$$

$$\begin{aligned}\beta^{(2)}_{2(n+1,n+2)} &= |2\rangle_{n+1,n+2}\langle 1|_{n+1}\langle\xi_{\mathcal{N}}|_{n+2} + |2\rangle_{n+1,n+2} \\ &\times\sum_{i=1}^{\mathcal{N}}\langle\mathcal{F}_i|_{n+2}\langle\xi_{\mathcal{N}-i}|_{n+1} + \bar{\rho}_{n+1}\otimes|1\rangle_{n+2}\langle\xi_{\mathcal{N}}|_{n+1} \\ &+\frac{1}{\mathcal{Z}_1}(|\mathcal{F}_1\rangle_{n+1}|1\rangle_{n+2}\langle 1|_{n+1}\sum_{i=1}^{\mathcal{N}}\langle\mathcal{F}_i|_{n+2}\mathcal{Y}^1_{\mathcal{N}-1}), \quad (11)\end{aligned}$$

where $\mathcal{Y}^1_{\mathcal{N}} = \langle\mathcal{N}|\mathcal{N}-1\rangle|1\rangle$. The states $|\mathcal{F}_i\rangle$ can be recursively generated, as shown below. To obtain the reduced density matrix for a large spin system, the parameters that appear upon tracing must be recursively generated from the parameters corresponding to smaller systems that can be exactly calculated.

The different "inner products" can be calculated as follows. The recursion begins with evaluating $\langle 1|\bar{2}\rangle = \sum_{i=1}^{k}\alpha^1_i|\alpha_i\rangle$, with $\{|\alpha_i\rangle\}_i$ forming an independent set of vectors consisting of certain singlet combinations of an $(1,\mathcal{N}+2)$ spin system, where k has to be found explicitly. E.g., $|\alpha_1\rangle$=$|1\rangle$. In general, we can write

$${}_n\langle\alpha_j|\bar{2}\rangle_{n,n+1} = (-1)^{n-1}\sum_i\alpha^j_i|\alpha_i\rangle_{n+1}. \quad (12)$$

Using the above relation, we can generate the inner product recursions as follows:

$$\begin{aligned}\mathcal{Z}_{\mathcal{N}} &= \langle\mathcal{N}|\mathcal{N}\rangle_n \\ &= \mathcal{Z}_1\mathcal{Z}_{\mathcal{N}-1} + \mathcal{Z}'_2\mathcal{Z}_{\mathcal{N}-2} + 2(-1)^{n-1}\sum_i\alpha^1_i\mathcal{Y}^i_{\mathcal{N}-1}, \quad (13)\end{aligned}$$

$$\begin{aligned}\mathcal{Y}^j_{\mathcal{N}} &= {}_n\langle\alpha_j|(_{1,n-1}\langle\mathcal{N}-1|\mathcal{N}\rangle_{1,n}) \\ &= \mathcal{A}_{j1}\mathcal{Z}_{\mathcal{N}-1} + (-1)^{n-1}\sum_i\alpha^j_i\mathcal{Y}^i_{\mathcal{N}-1}, \quad (14)\end{aligned}$$

where $\mathcal{Z}'_2 = \langle\bar{2}|\bar{2}\rangle$ and $\mathcal{A}_{ij} = \langle\alpha_i|\alpha_j\rangle$. Similarly, the other states can be generated using subsequent recursions,

$$\langle\xi_{\mathcal{N}}|_{n+1} = \langle\bar{2}|_{n,n+1}\langle\mathcal{N}-1|\mathcal{N}\rangle = \sum_{i=1}^{\mathcal{N}}\mathcal{Z}_{\mathcal{N}-i}\langle\mathcal{F}_i|_{n+1}, \quad (15)$$

where $\langle\mathcal{F}_i|_{n+1} = {}_{n,n+1}\langle\bar{2}|\mathcal{F}_{i-1}\rangle_n$ and $|\mathcal{F}_0\rangle_n = |1\rangle_n$. On expanding, we can write

$$\langle\mathcal{F}_i|_{n+1} = {}_{n,n+1}\langle\bar{2}|\mathcal{F}_{i-1}\rangle_n = \sum_k g^i_k\langle\alpha_k|_{n+1}, \quad (16)$$

where $g^i_j = \sum_k g^{i-1}_k\alpha^k_j$. Hence all the terms can be recursively calculated provided the paramters for small spin systems can be accurately estimated. The terms $\alpha^j_i(i,j =$ 1 to $k)$, of $\mathcal{Z}_1$, $\mathcal{Z}'_2$ and $\mathcal{A}_{ij}$ need to be exactly calculated. The value of k needs to be determined for a small system size by solving the linear equation system (12).

B. Imperfect Square Lattice

Let us now consider spin lattices with M sites on the horizontal side and M' on the vertical one, where M is even, while M' is odd. Here $M.M'$ (=$2N$) and M' (=$M\pm1$). For a system containing $M=\mathcal{N}+2$ spin sites along the horizontal side, the recursion of the periodic state $|\mathcal{N}+2,\mathcal{N}+2\pm1\rangle_P$ can be written as

$$|\mathcal{N}+2\rangle_P = |\mathcal{N}\rangle_{1,n}|2\rangle_{n+1,n+2} + |\mathcal{N}\rangle_{2,n+1}|2\rangle_{n+2,1}. \quad (17)$$

Here, the constant $M' = \mathcal{N}+2\pm1$ has been omitted from the notation of the state. $|2\rangle_{i,j}$ refers to the state corresponding to $(M,M') = (2,\mathcal{N}+2\pm1)$.

The reduced density matrices can be recursed by using Eq. (17). In particular, for obtaining a reduced density matrix of the spins at columns $(n+1,n+2)$, we trace out the spins in the columns ranging from 1 to n, and obtain

$$\begin{aligned}&\rho^{(2)}_{P(n+1,n+2)} = \mathrm{tr}_{1..n}[|\mathcal{N}+2\rangle\langle\mathcal{N}+2|_P] \\ &= \mathrm{tr}_{1..n}[|\mathcal{N}\rangle\langle\mathcal{N}|_{1,n}|2\rangle\langle 2|_{n+1,n+2} + |\mathcal{N}\rangle\langle\mathcal{N}|_{2,n+1}|2\rangle\langle 2|_{n+2,1} \\ &+(|\mathcal{N}\rangle_{1,n}|2\rangle_{n+1,n+2}\langle\mathcal{N}|_{2,n+1}\langle 2|_{1,n+2} + h.c.)]. \quad (18)\end{aligned}$$

After simplification, the above equation reads

$$\begin{aligned}\rho^{(2)}_{P(n+1,n+2)} &= \mathcal{Z}_{\mathcal{N}}|2\rangle\langle 2|_{(n+1,n+2)} + \mathcal{Z}_{\mathcal{N}-2}\bar{\rho}_{n+1}\otimes\bar{\rho}_{n+2} \\ &\quad + (|2\rangle_{n+1,n+2}\langle\chi_{\mathcal{N}}|_{n+1,n+2} + h.c.), \quad (19)\end{aligned}$$

where the normalization is defined as $\mathcal{Z}_{\mathcal{N}} = \langle\mathcal{N}|\mathcal{N}\rangle = \mathcal{Z}_2^{N/2}$. The recursion for the state $|\chi_{\mathcal{N}}\rangle$ is given by

$$\begin{aligned}\langle\chi_{\mathcal{N}}|_{n+1,n+2} &= \langle 2|_{1,n+2}\langle\mathcal{N}|_{2,n+1}|\mathcal{N}\rangle_{1,n} \\ &= \langle 2|_{1,n+2}\langle 2|_{2,3}...\langle 2|_{n,n+1}|2\rangle_{1,2}|2\rangle_{3,4}...|2\rangle_{n-1,n}. \quad (20)\end{aligned}$$

The last recursion can be done numerically to solve for $\langle 2|_{i,i+3}\langle 2|_{i+1,i+2}|2\rangle_{i,i+1}$.

In the succeeding subsection, finite size calculations shows that the behavior of multisite entanglement of the two types of lattices merge as the size of the system increases.

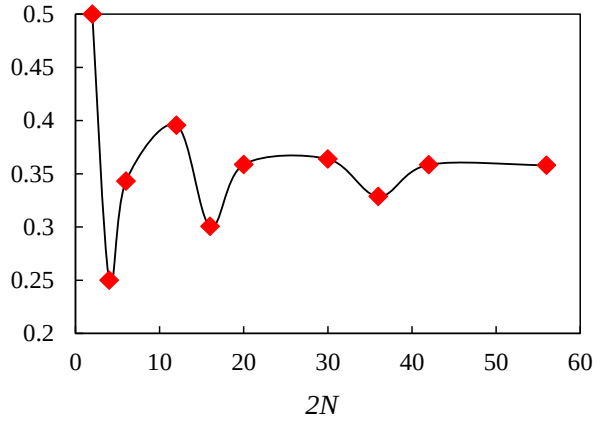

FIG. 2. (Color online.) The behavior of GGM in the case of a square spin-1/2 lattice, with increasing total number of spins ($2N$).

C. Finite size scaling

We are now ready to calculate the genuine multipartite entanglement of spin-1/2 square lattices by using the DMRM for such lattices. We find the GGM for perfect as well as imperfect square lattices and observe that for increasing system size, the GGM converges to the value 0.358.

In Fig. 2, we plot the GGM with increasing total number of spins ($2N$). Using finite size scaling, the behavior of GGM for finite-sized lattices can be used to estimate the GGM for an infinite square lattice. The scaling analysis gives

$$G(|\psi\rangle) \approx G_c(|\psi\rangle) \pm k\ \mathbb{N}^{-x} \tag{21}$$

where $\mathbb{N} = 2N$ and $G_c(|\psi\rangle)$ is an estimated value of GGM for the infinite lattice, based on the average of the last two values of GGM given in Fig. 2, with k being a constant. The value of x, as estimated by finite-size scaling, using $G_c(|\phi_N\rangle)$=0.358, is x =1.69, while k=0.843.

V. DISCUSSIONS

We investigate multipartite entanglement of a large superposed state consisting of dimer coverings of a spin-1/2 isotropic lattice. We first showed that the state, if isotropic over the lattice, with either periodic boundary conditions or is an infinite lattice, is genuinely multipartite entangled, regardless of the geometry and dimension of the isotropic lattice. To measure its multisite entanglement content, we have presented a technique, which we have referred to as the density matrix recursion method for square lattices, to analytically study arbitrary state parameters of an arbitrary number of sites, including its genuine multisite entanglement. The method was then employed to investigate the finite-size scaling behavior of the generalized geometric merasure, which is a measure of genuine multipartite entanglement, for finite-sized square lattices that enables us to to estimate the genuine multipartite entanglement of an infinite square lattice.

Genuine multisite entanglement is potentially a basic ingredient in building large-scale quantum computers and also in implemntation of multiparty quantum communication. The method presented can be a useful tool if such highly superposed systems are considered for performing quantum tasks. Specifically, the iterative method can be employed to derive reduced density matrices that will in turn be fruitful in the calculation of nearest-neighbor bipartite entanglement as well as that of other two-point correlation functions.

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